Stochastic Processes

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2006

1 Motivation

We use stochastic processes to develop important concepts in finance, the martingale (describes an efficient market under risk neutrality), Markov processes ("memoryless" property) and Brownian motion (used in stochastic calculus and option pricing).

2 A note on probability

In distinct contrast from the various "real world" interpretations of probability (classical, frequency, propensity, logical (Bayesian) and subjective (Bayesian)) we use the mathematical probability theory of pure mathematics. The theory was developed by Kolmogorov (1933) and is part of measure theory and more generally analysis.

3 Definitions

Definition 1 *Let A be a set. $\mathcal{F}$ is a* sigma algebra (or *σ-algebra) if and only if:*

- $\emptyset \in \mathcal{F}$ *and* $A \in \mathcal{F}$
- *If* $A \in \mathcal{F}$ *then* $A^c \in \mathcal{F}$
- *If* $A_i \in \mathcal{F}$ *for* $i = 1, 2, \ldots$ *then* $\bigcup_i A_i \in \mathcal{F}$.

Intuitively, the collection must include any result of complementations, unions, and intersections of its elements. The effect is to define properties of a collection of sets such that one can define probability on them in a consistent way.

Definition 2 Borel algebra (or *Borel σ-algebra) on a topological space X is the minimal σ-algebra containing the open sets of X.*

Definition 3 *A* probability space is a triple, $(\Omega, \mathcal{F}, P)$, where

- Ω is the *sample space of all possible outcomes*

- $\mathcal{F}$ *is the* event space. Its elements are subsets of Ω, and it is required to be a σ-algebra

- P is a measure on $\mathcal{F}$ with $P(\Omega) = 1$ which is known as the *probability measure*

Definition 4 *A state space* $(E, \mathcal{E})$ *is a measurable space.*

Definition 5 *A* random variable is a measurable function from a probability space to some measurable space.

Definition 6 *A stochastic process is a family of random variables* $(X_t)_{t \in I}$ *from some probability space* $(\Omega, \mathcal{F}, P)$ *into a state space* $(E, \mathcal{E})$*. The set* I *is the index set.*

- *discrete time, discrete state space*

- *discrete time, continuous state space*

- *continuous time, discrete state space*

- *continuous time, continuous state space*

The prices of tradable assets are restricted to discrete values and changes can be observed only when the market is open. However, the continuous time, continuous variable process proves to be the most useful for the purposes of valuing derivative securities.

Definition 7 *Given a measurable space* $(E, \mathcal{E})$*, a* filtration is a sequence of sigma-algebras $F_t : 0 < t < \inf$ with F_t contained in F for each t.

Definition 8 *A* martingale is a discrete-time stochastic process that satisfies the equality:
$E[X_{n+1}|X_1, \ldots, X_n] = X_n$

Definition 9 *A* submartingale is a discrete-time stochastic process that satisfies the inequality:
$E[X_{n+1}|X_1, \ldots, X_n] \geq X_n$

Definition 10 *A* supermartingale is a discrete-time stochastic process that satisfies the inequality:
$E[X_{n+1}|X_1, \ldots, X_n] \leq X_n$

Definition 11 *A* Markov process is a stochastic process which satisfies the equality:
$P(c_k|c_0, c_1, \ldots, c_k - 1) = P(c_k|c_{k-1})$

Definition 12 *A* Lévy process is any continuous-time stochastic process that has “stationary independent increments” The increments of such a process are the differences $X_s - X_t$ between its values at different times $t < s$.

Definition 13 *A* Gaussian process is a stochastic process $\{X_t\}_{t \in T}$ such that every finite linear combination of the X_t is normally distributed.

Definition 14 *A* diffusion process is a stochastic process with independent increments where the 'displacement' of the variate (its increment) in time dt follows a normal distribution with variance proportional to dt.

Definition 15 Wiener process/*Brownian motion is a continuous-time Gaussian stochastic process with independent increments. The probability distribution of* $X_s - X_t$ *is normal with expected value 0 and variance* $s - t$.

Definition 16 random walk is the discrete equivalent of Brownian motion. That is, Brownian motion is the scaling limit of random walk in dimension 1.

4 Applications

Which of the above (if any) apply to real financial markets?

If investors are risk-neutral, markets follow a martingale. In practice, of course, investors are not risk neutral, so the expected future price is always greater than or equal to the current value (presumably as compensation for the time value of money and systematic risk)so that the price of a stock follows a submartingale.

In a Markov process the conditional probability distribution of the future price depends only on the current price. This is a very good approximation to real markets.

Brownian motion is a very strong condition, and not a good approximation of real financial markets.

However, Brownian motion is a martingale, a Markov process, a Gaussian process, a diffusion, and a Lévy process, so is a very important and "generic" stochastic process which is used extensively in stochastic calculus for pricing options.