Decision Making with Imprecise Probabilities

Brian Weatherson

Abstract

Orthodox Bayesian decision theory requires an agent's beliefs representable by a real-valued function, ideally a probability function. Many theorists have argued this is too restrictive; it can be perfectly reasonable to have indeterminate degrees of belief. So doxastic states are ideally representable by a set of probability functions. One consequence of this is that the expected value of a gamble will be imprecise. This paper looks at the attempts to extend Bayesian decision theory to deal with such cases, and concludes that all proposals advanced thus far have been incoherent. A more modest, but coherent, alternative is proposed.

Keywords: Imprecise probabilities, Arrow's theorem.

1. Introduction

Orthodox Bayesian decision theory requires agents' doxastic states to be represented by a probability function, the so-called 'subjective probabilities', and their desires to be represented by a real-valued utility function. Once these idealisations are in place, decision theory becomes relatively straightforward. The best choice is the one with the highest expected utility according to the probability function. Because of Newcomb-like problems there is little consensus on how we ought to formalise 'expected utility according to a probability function', but in the vast bulk of cases the different approaches will yield equivalent results.

The main problem for orthodoxy is that the idealisations made at the start are highly questionable. Many writers have thought that it is no requirement of rationality that agent's epistemic states be representable by a single probability function. Others have thought that even if this is an ideal, it is so demanding that we cannot expect humans to reach it. One attractive amendment to orthodoxy is to permit agents's epistemic state to be represented by a set of probability functions. This idea was first suggested by two economists, Gerhard Tintner (1941) and A. G. Hart (1942). It has since been rediscovered and popularised by Smith (1961), Levi (1974, 1980), Williams (1976, 1978), Jeffrey (1983) and van Fraassen (1990, 1995). An almost identical proposal is worked out in great detail in Walley (1991). There are many motivations for this, not least of which are that it allows agents to be completely represented by a finite number of constraints and it allows a consistent representation of ignorance. The set of probability functions representing an agent's epistemic state is conveniently called her *representor*. We say that an agent's degree of belief in p is vague over the set of values $Pr(p)$ takes for each element Pr of her representor.

Once we make this amendment, however, our neat decision theory vanishes. Even assuming Newcomb-like problems to be resolved, all expected utility calculations tell us is the utility of each decision according to each probability function. In other words, different functions in a representor will usually produce different expected utilities for a choice. So the expected utility of an action is not a number, but a set. For simplicity, I will assume that these sets form intervals; on most of the theories mentioned above this follows from the way representors are constructed. I will also assume, somewhat arbitrarily, that the sets are closed intervals; nothing turns on this and it does simplify the presentation. The important point is that these intervals may overlap. When they do, what ought an agent choose?

This question has been addressed by many authors, as will be clear from the discussions below, but none have provided a satisfactory answer. Much of the discussion has taken place in the economics literature, so the focus has been on trades. This is of more than cosmetic importance. It has meant that the decision situation discussed contains a crucial asymmetry. Because there is a default position, refraining from trade, we can formulate clear distinction between acts and omissions. This distinction can be incorporanted into our decision theory. I don't think theories based on this distinction eventually work, but I don't believe their use of this asymmetry is sufficient to refute them.

So with this background, the central question is: given an agent's representor, when should she trade ϕ for ψ? This divides into two questions: when is trade permissible, and when is it rationally required? I will also be interested in some associated questions, such as determining which choices are permissible (or mandatory) from a set of available decisions. As noted, the 'central question' contains a deliberate asymmetry between ϕ and ψ.

I'll define an A-bet, or a bet on A, as a bet which pays \$1 if A and nothing otherwise. And as is standard I'll assume the marginal utility of money is constant, so the value, according to a probability function *Pr*, of an A-bet is $Pr(A)$. I'll restrict my formal discussion to cases when ϕ and ψ are gambles or bets, as Ramsey (1926) points out these are quite representative of the decisions we make in everyday life.

For a bet ϕ, an agent's representor P will determine a range of *expected* values for ϕ: $[l_\phi, u_\phi]$. It's very important to remember that $[l_\phi, u_\phi]$ is not the range of possible payouts for ϕ; that range will usually be considerably wider, and need not be an interval. I am not interested in what the agent thinks ϕ might pay, rather in, roughly, what she thinks ϕ can be expected to pay. If her degrees of belief are all precise so her representor is a singleton, then, whatever the range of payouts of ϕ, l_ϕ will equal u_ϕ.

Decision theories which allow for imprecise credences fall into two broad categories: structured and unstructured. Unstructured decision theories say we can determine the relative merits of ϕ and ψ by just looking at l_ϕ, u_ϕ, l_ψ and u_ψ. Structured decision theories say we need to look at more; in particular, we need to compare the values ϕ and ψ according to particular members of P. The first three theories I'll look at are unstructured; it can be concluded from the way they fail that no unstructured decision theory is plausible.

The bulk of this paper is negative; I show why a glut of solutions to our problem given in the literature fail. Many of the refutations rely on a rather odd epistemic state, one where an agent has a precise degree of belief in p, $Pr(p)$ is the same for each Pr in P, but she knows that some information will come in such that, whatever it is, she will have an imprecise degree of belief in it. The latter means, of course, that for some different Pr in P, $Pr(p)$ will be different. If we take her 'uncertainty' about p to be measured by the range of values that $Pr(p)$ takes for $Pr \in P$, this means that finding out the result of an experiment can ensure that an agent because less certain about p, whatever the experiment says. As these states cause problems for many of the theories which follow, it might be wondered whether such odd states can be ruled out as unreasonable.

The answer is they cannot, at least on pain of ruling out all imprecise states as unreasonable. Seidenfeld (1994) shows that on some simple assumptions[1], the requirement that states be immune to what he calls *dilation* is equivalent to the requirement that states be precise. Let P be a set of probability functions, and let $min(h \mid e)$ and $max(h \mid e)$ be defined as the minimal and maximal values respectively of $Pr(h \mid e)$ for $Pr \in P$. Let $\Pi = \{p_1, ..., p_n\}$ be a partition of e. That is, the elements of Π are pairwise disjoint, and their disjunction is e. Then P is dilated by Π with respect to h and e if for all i, $min(h \mid p_i \,\&\, e) < min(h \mid e)$ and $max(h \mid p_i \,\&\, e) > max(h \mid e)$. A set P is subject to dilation if there is a h, e, and Π such that P is dilated by Π with respect to h and e. Since requiring that P be immune to dilation amounts to insisting that P be a singleton (or satisfy some other even more implausible constraints so that Seidenfeld's assumptions fail) I don't think that requirement can be plausibly imposed. Hence we must learn to live with the decision theoretic consequences of dilation.

I will assume that even though we have allowed representors to contain multiple probability functions, they are still updated by conditionalisation. Arguments for this can be found in van Fraassen (1990) and Walley (1991). So if an agent's representor is P, and she learns E, her new representor is:

$$\{Pr: \exists Pr' \in P \,\&\, Pr = Pr'(\bullet \mid E)\}$$

I will also be using a restricted principle of conglomerability, as set out immediately below.

Restricted Conglomerability

Let ϕ and ψ be bets, such that it would be rationally mandatory for an agent to trade ϕ for ψ were she to learn A, or were she to learn $\neg A$. Then it is rationally mandatory for the agent to trade ϕ for ψ.

Now some theorists deny even Restricted Conglomerability because of Prisoners Dilemma and Newcomb Problem cases. I don't want to get into this debate, so I'll just note that I won't use the rule in a way that ought offend such theorists.[2]

In any case, this isn't the objection which concerns me most. What does concern me is the possibility that two-envelope paradox type cases show that conglomerability is implausible. The principle I've stated is that if ϕ is better than ψ according to every member of a finite partition, it is better *simpliciter*. As Arntzenius and McCarthy (1997) show, the principle becomes incoherent if we replace 'finite' with 'countable'. Perhaps this shows conglomerability is generally implausible. But perhaps this is just guilt by association. So while Restricted Conglomerability is not an entirely safe assumption, it has a very high intuitive plausibility, and until there is a solid argument against it I will continue to use it.

---

[1] The needed assumption is basically just that there are some probabilistically independent propositions.

[2] As a rule like conglomerability is usually needed to justify the idea that we ought to value gambles by their expected utility, objections to it on the ground that it conflicts with the verdict of utility considerations in Newcomblike cases seem to me implausible. Recently, Norton (1998) has argued that we shouldn't accept the verdict of conglomerability in the two-envelope paradox because it conflicts with expected utility considerations. Well, he's right that we shouldn't accept all of conglomerability's verdicts here, but that's because they are inconsistent, not because of the clash with expected utility. Again, without some form of conglomerability there is no motivation for adopting a rule like 'maximise expected utility'.

2. Unstructured Decision Theories

2.1. Global Dominance

Hájek (to appear) discusses without endorsing a decision rule called global dominance. This says that it is only rationally compelling to trade ϕ for ψ when $l_\psi > u_\phi$. It isn't made clear, but presumably whenever $u_\psi > l_\phi$ it is rationally permissible to trade. There is a simple counterexample to this approach. Let ψ be the bet ϕ + $\$\varepsilon$, where ε is some small amount of money such that $l_\phi + \varepsilon = l_\psi < u_\phi$. That is, in any circumstance where ϕ pays $\$m$, ψ pays $\$(m + \varepsilon)$. Clearly here it is rationally compelling to trade ϕ for ψ, however the global dominance rule does not require this.

2.2. Maximin

Gilboa and Schmeidler (1993) advocate a maximin decision rule. The rule is that it is rationally compelling to trade iff $l_\psi > l_\phi$, and rationally permissible to trade iff $l_\psi \geq l_\phi$. While this rule doesn't give any particularly counterintuitive results for static cases, it seems to do badly in dynamic settings. Their rule wasn't designed to be used with conditionalisation, so the objection I'm running isn't directed at any particular theorist, just at its possible use with the Bayesian updating rule.

Let $P = \{Pr: Pr(q) = 0.5\}$. An agent represented by P and following Maximin will gladly buy a q-bet for 40 cents according to the maximin rule. Assume this trade is made. Now were the agent to learn either p and $\neg p$, their degree of belief in q would become vague over [0, 1], since in P there are probability functions Pr and Pr' such that $Pr(q \mid p) = z = Pr'(q \mid \neg p)$ for every z in [0, 1]. Hence after learning either p or $\neg p$ they will sell this bet for 20 cents, or indeed for any positive amount, thus incurring a sure loss. This is inconsistent with restricted conglomerability, so the agent is incoherent.

2.3. Maxi

This problem could be avoided by adopting a decision rule I call *Maxi*. This says that ψ is strictly preferred to ϕ, i.e. trade is rationally compelling, iff $l_\psi > l_\phi$ and $u_\psi > u_\phi$. Trade is rationally permissible iff $l_\psi \geq l_\phi$ or $u_\psi \geq u_\phi$. No one to my knowledge has endorsed *Maxi* in the literature, but since it is such an obvious weakening of Maximin and other such rules which have been endorsed, it is worth some discussion.

Although there are no simple examples where Maxi gives counterintuitive results, it is in conflict with conglomerability in some hoked-up examples. If one was committed to Maxi, I suppose it could be said that these were arguments against Restricted Conglomerability rather than Maxi; or alternatively, that in such bizarre examples we can't expect standard rules to apply. I don't think either of these replies works, but I mention them to note that my objections to Maxi are weaker than my objections to other rules.

Say an agent's degrees of belief are determined by the family of probability functions satisfying the following criteria:

(i) $0.2 \leq Pr(p \mid r) \leq 0.6$

(ii) $0.1 \leq Pr(q \mid r) \leq 0.5$

(iii) $0.3 \leq Pr(p \mid \neg r) \leq 0.7$

(iv) $0.2 \leq Pr(q \mid \neg r) \leq 0.6$

(v) $Pr(p) = 0.35$

(vi) $Pr(q) = 0.4$

It can quickly be seen that none of these conditions are redundant by considering functions like Pr_1, defined as follows. $Pr_1(p \mid r) = 0.2$; $Pr(p \mid \neg r) = 0.7$, $Pr(r) = 0.7$, $Pr(q \mid r) = Pr(q \mid \neg r) = 0.4$. Similar functions show the other six bounds given in the inequalities are non-redundant. Given this epistemic state the value of a *p*-bet will be precisely 35 cents, and the value of a *q*-bet precisely 40 cents. However, if the agent were to discover *r*, the value (in dollars) of a *p*-bet would be vague over the interval [0.2, 0.6], and that of a *q*-bet vague over [0.1, 0.5]; that is a *p*-bet would be more valuable, according to Maxi, were the agent to discover *r*. Similarly if the agent were to discover $\neg r$, the value of a *p*-bet would go to [0.3, 0.7] and of a *q*-bet would go to [0.2, 0.6]. Again by Maxi, the *p*-bet would be more valuable.

Hence in these circumstances, Maxi gives the result that a *q*-bet is more valuable than a *p*-bet (by 5 cents), however if either *r* or $\neg r$ were found to be true, it would become the case that a *p*-bet would be 10 cents more valuable than a *q*-bet. That is, Maxi is in breach of Restricted Conglomerability. Given that the problem with Maxi is that it is too strong, in the sense that it cannot be that all of the trades which are rationally compelling according to Maxi are really compelling we can draw a more important conclusion. There is no rule expressed purely in terms of l_ϕ, u_ϕ, l_ψ and u_ψ which is stronger than Global Dominance but weaker than Maxi. Yet I've shown that any acceptable rule must be stronger than Global Dominance and weaker than Maxi. Hence no acceptable rule can be expressed purely in terms of l_ϕ, u_ϕ, l_ψ and u_ψ.

As a special case, the Horvitz-style decision rules advocated by Strat (1990) and Jaffray (1994) are incoherent. These advocate that for any bet ϕ we evaluate its expected worth $E(\phi)$ according to this rule.

$$E(\phi) = \rho l_\phi + (1 - \rho)u_\phi.\ (\rho \in [0, 1]).$$

The operator ρ is an optimism / pessimism operator. The more optimistic we are the higher ρ will be. Since we now have a numerical utility for each bet, we can simply choose the bet with the higher utility. Of course this approach is stronger than Maxi, so if Maxi is too strong, so is this approach. Here the fact that the counterexamples to Maxi are so artificial becomes important, because Strat and Jaffray are not, it appears, aiming to discover the ideal decision rule, but rather trying to find a rule which can be implemented efficiently and gives results which are usually correct. Until an example is found in which the recommendations of this approach are implausible despite the example being realistic enough, their approach might be well-suited to the task they have set themselves.

3. Levi's Rule

For the subsequent rules I'll be discussing, I need to look more closely at the structure of the expectation values, not just at their upper and lower bounds. For any bet, say ϕ, and any element Pr of P, there is a numerical expectation value of ϕ, which we'll call $E_{Pr}(\phi)$. All the subsequent rules I discuss have the property that if for all

Pr in *P* $E_{Pr}(\psi) > E_{Pr}(\phi)$, then ψ is strictly preferred to ϕ. That is, it is rationally compelling to trade ϕ for ψ. How the rules differ is in what can be done when neither bet is strictly preferred to the other in this sense. For convenience, I'll simply define strict preference to hold between two bets ψ and ϕ iff $E_{Pr}(\psi) > E_{Pr}(\phi)$ for all *Pr* in *P*. This reduces the scope of discussion to bets such that neither is strictly preferred to the other. I will say in this case that the bets are *almost indifferent*. On pain of inconsistency it can't be said that almost indifference implies indifference. This is because almost indifference is intransitive whereas indifference, at least as usually defined, in transitive.

Levi's Rule is that when ϕ and ψ are almost indifferent we should choose the bet which has the highest minimum payout (Levi 1974, 1980, 1986). This minimum payout is referred to as the 'security level' of the bet. I'm keeping with Levi's terminology in referring to choices rather than permissible trades; the translation back into terminology I've been using is usually trivial. He doesn't mean by this that we ought choose ψ iff $l_\psi > l_\phi$. Rather he is referring back to the actual payouts of ϕ and ψ and advocating choice of the bet with the highest possible minimum return. As he notes, when applied to three-way choice this implies violation of the rule of independence of irrelevant alternatives. That is, under his rule it can be rationally required that ϕ be chosen in a pair-wise choice from {ϕ, ψ}, but also required that ψ be chosen in a choice from the set {ϕ, ψ, χ}. Since he regards the analysis he offers as "impeccable" (1974: 411) he concludes that the rule of independence of irrelevant alternatives must be mistaken in some way.

It's not too surprising that this rule would have to go under such an analysis. After all we can regard each of the *Pr* as a voter which voices an opinion about which choice is best, and then the overall choice becomes the well-known social choice problem of aggregating preferences. Arrow's theorem says that no aggregation rule can satisfy the following four constraints, here explained for voters who are probability functions[3]:

(1) *Pareto*. If ϕ is strictly preferred to ψ in the above sense ϕ will be chosen from {ϕ, ψ}.

(2) *Collective Rationality*. The rule determines a preferred option no matter what the various *Pr* functions say about ϕ and ψ.

(3) *Non-Dictatorship*. There is no *Pr* function whose choice is followed no matter what the other functions say.

(4) *Independence of Irrelevant Alternatives*. The choice between ϕ and ψ should not depend on what other options are available.

Levi's Rule is committed to (1), (2) and (3), hence it would be inconsistent if it satisfied (4). However, there are good grounds for preserving (4). Of course, there are good grounds for keeping each of these rules, so this argument will necessarily be less than completely compelling. I suspect the strongest argument for (4) is its intuitive plausibility; any attempt to explain this plausibility will sell it short. Nevertheless, I'll try.

Assume an agent, say Lenny, does not satisfy (4). For example, he chooses ϕ from {ϕ, ψ}, but chooses ψ from {ϕ, ψ, χ}. Assume now he has a choice between {ϕ, ψ, χ}, but the choice dynamics are as follows. First, he has to specify whether he wants χ or not, and if not he has to say whether he wants ϕ or ψ. Lenny's preference

[3] Arrow's Theorem is set out in Arrow (1963). The setting out here closely follows Hausman (1991).

is, *ex hypothesi*, to choose ψ, but he can't carry out this choice. Presumably he will reject χ at the first stage, then he will face a choice between ϕ and ψ. And here he is forced by his own preferences to choose ϕ. Levi (1987) in response to this argument claims that Lenny could have adopted at the start a strategy to choose ψ. Hence, at the second stage he will just have to follow his strategy rather than to make a decision about whether ϕ or ψ is preferable. But now the original objection can be restated in a different way. Surely it's a problem for a decision-rule if the only way to consistently implement it in a dynamic context is to ignore its recommendations at various stages. Alternatively, it might be argued that the amendment to the rule to allow strategic choice in this way constitutes a rescission of the original rule and substitution of a new rule. The basis for this argument is simply that, according to the amended rule, at times agents times are required to act in the opposite way to how they were required to act under the old rule.

Levi tries to minimise this difficulty by saying that it is an ineliminable feature of what he calls 'unresolved conflict'. The problem is that he seems to rely here on some equivocations about what would count as a resolution of a conflict. This leads to a problem at the core of his lexicographic approach. Levi thinks that we can have a hierachy of 'values', such that if we can't decide between two options using the most important value, we can use lower values to resolve it. That's essentially what is being applied here, with expected value being the highest value, and security levels the next. When it is allowed that each of these values might issue non-linear verdicts (they might allow us to be unresolved and not just indifferent between choices) the lexicographic approach hits problems. The problem is essentially that he seems to be commited to saying that some decision making contexts involve a conflict which is essentially unresolved, while at the same time saying that there is a resolution of these conflicts!

Here's an example he gives. Jones, an office manager, has to hire a new worker to do typing and stenographical work. There are three applicants: Jane, Dolly and Lilly. The applicants take tests in typing and stenography. On the typing test their scores are 100, 91 and 90 respectively, on the stenography test the scores are 90, 91 and 100. So Jones has a dilemma; does he hire the best typist, or the best stenographer, or perhaps someone moderately good at each?

Levi suggests that there are in fact a continuum of tests Jones could apply. For each $\beta \in [0, 1]$ we can work out a candidate's β-score as $\beta x + (1 - \beta)y$, where x is their typing test score and y their stenography score. For each β test there corresponds an argument for selecting the applicant with the highest score on that test. These arguments will often conflict, as in fact they do here. Some tests favour Jane, and some favour Lilly. Since, however, none favour Dolly she can be ruled out. Now Jones is a liberal, but to a degree: he favours using affirmative action criteria to choose a candidate when the continuum of β-tests have failed to be decisive. The affirmative action criteria support ranking the applicants as follows: Dolly, Jane and Lilly. Since Jane is the highest ranked of the candidates left (not ruled out by the β-tests), she gets the job.

But there's a twist to the tale. Just as he's about to tell Jane she has the job, he finds Lilly has withdrawn her application. Now he has to choose between just Jane and Dolly. And since on some β-tests Dolly is now the best of the applicants (where $\beta < 0.1$) she isn't ruled out by those tests. Hence Jones has to make a decision between Jane and Dolly on affirmative action grounds, and *ex hypothesi* Dolly wins. So Lilly's withdrawl means that Dolly now gets the job over Jane.

Levi notes that most decision theorists would demur here. After all, Jones, a poster-boy for his decision-theory, has just violated what we're calling independence of irrelevant alternatives. Here's his defence:

> When Jones chooses Dolly, this does not reveal that he thinks Dolly is at least as good as Jane for the job. Jones is in conflict as to who is better, all things considered. He chooses Dolly because in the face of such conflict among the values to which he is committed, he invokes considerations which otherwise would not have counted for him. When he contemplates the three-way choice, hiring Dolly is ruled out because of his values. This does not mean that his values have changed or that he has inconsistent values. *Hiring Dolly is neither better nor worse than hiring Jane in the two-way choice.* The same remains true in the three-way choice. This example illustrates an important difference between resolving a conflict so that one can choose for the best and failing to resolve a conflict. In the latter case, some consideration which otherwise would not be taken into account is used to provide counsel as to what to do when one cannot choose for the best, all things considered. (1986: 34, my italics)

But this is inconsistent with his description of Jones's motivation. Jones has made a value commitment to hiring on the basis of affirmative action when the β-tests are inconclusive. This is why it can be deduced from his general principles (including his 'tie-breaker' principles) that he will hire Dolly. It is worse, given his principles, to hire Jane over Dolly in the two-way choice, contra what is said in the italicised sentence. Levi wants here to have it both ways; Jones's affirmative action commitment is supposed not to be a value of any kind, so that it wouldn't be against his values to hire Jane over Dolly, but it can at the same time be used 'to provide counsel as to what to do'. It is rather hard to see how this is consistent. To paraphrase Ramsey, if Jones can't say what his choice is, he can't say it, and he can't whistle it either. If two options really are incommensurable, there can't be a reason for choosing one over the other; that just would show that they weren't really incommensurable to start with.

As a footnote to all this, at (1986: 82) he says that rational agents may have a hierachy of value commitments. This seems to suggest that he favours saying Jones's commitment to affirmative action is a value, in which case the italicised sentence is simply false, so his general defence here fails.

More difficulties can be made for Levi's decision theory. Assume we have the followng test scores for some new applicants.

	Typing	Stenography
Tom	100	90
Dick	90	100
Harry	89	99

We have the following affirmative action ordering: Harry, Tom, Dick. If we adopt Levi's rule, we will choose Tom for the position. Dick's scores dominate Harry, so Harry can't pass any of the β-tests. However, both Tom and Dick pass some, so the affirmative action test applies, and Tom is chosen. Now assume that instead of choosing one applicant for a position, we have to choose two. We could assume that Tom will be chosen, leaving a two-way choice between Dick and Harry for the final position, which presumbly goes to Dick.

It might be thought more efficient, however, to decide whom it would be worst to give the position, and hence offer jobs to the other two. The only plausible way to do this is simply to reverse our tests. So at the first stage we'll look at who's worst on all β-tests, as this is our main criteria. If there is more than one person who is worst according to some β-test, we'll look at who does worst by the affirmative action criteria among these. If we apply this method we find that the worst person to give the job to would be Tom! The only people who are worst according to some β-test are Tom and Harry, and Tom is further down the affirmative action list than Harry. So there are two absurd results: the best person to give the job to is also the worst, and we get different results to the question of which two people we should hire depending on whether we look for the best two candidates or the worst. For the reasons indicated above, I am unimpressed with Levi's assertions that choices on the basis of 'tie-breaker' principles are not real preferences. In summary, not only does Levi's rule give counterintuitive results, it rests on a methodology which is suspect because of this equivocation.

4. Conservatism

The rule I am calling Conservatism is perhaps the dominant decision-theoretic rule amongst Bayesians who allow degrees of belief to be vague. For endorsements of it, see for example Williams (1976) or Seidenfeld (1984) and the references contained therein. The rule is that it's rationally permissible to trade ϕ for ψ iff ψ is strictly preferred to ϕ. As noted above, the rule is asymmetric. There are circumstances in which it is impermissible to trade ϕ for ψ, and impermissible to trade ψ for ϕ. This is an oddity but not an inconsistency. If it was the worst that could be said for the rule it wouldn't be much of an objection. There is, however, a stronger objection.

Assume a Conservative is holding ϕ, and ψ is a bet which is almost indifferent to ϕ. Further assume that ϕ + \$10 is strictly preferred to ψ. The following is a simple-minded objection to Conservatism which doesn't work; I include it to distinguish it from an objection which does work. Assume the only trades which are possible are to swap ϕ for ψ, and, if that swap is made, to swap ψ for ϕ + \$10. It would clearly be in the agent's best interests to make each of these swaps, but since they are a Conservative they can't make the first swap, hence Conservatism is an irrational rule. The decision-tree is set out in Figure 1.

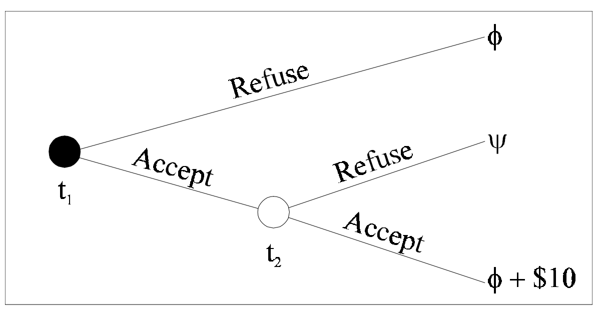

Figure 1

Here's what goes wrong with this objection. When considering the first swap, the Conservative won't be comparing ϕ and ψ; rather they will be comparing holding ϕ with the possibility of having a choice between having ψ and having ϕ + \$10. If they had the latter choice, they would choose ϕ + \$10, hence the original choice is between ϕ and ϕ + \$10. That isn't much of a choice at all, they will clearly choose the ϕ + \$10. That is, it is consistent with the Conservative rule to accept both trades.

So this objection fails because it relied on a too simplistic Conservative rule. However, a similar objection can succeed. Alter the payout of accepting both trades to ϕ + \$5, and assume this is strictly preferred to

ϕ, but almost indifferent to ψ. Now the initial choice is a choice between holding on to ϕ, and having the choice between holding ψ or trading it for ϕ + \$5. The Conservative knows if they have that choice they will hold onto ψ. So now the initial choice reduces to a choice between holding ϕ and trading it for ψ. Again, the Conservative here prefers to hold ϕ. But this is absurd. Whatever we should end up with in this circumstance, it isn't ϕ, as there is some other option strictly preferred to it. It might be noted that the use of decision-trees in this argument, as opposed to the flawed argument given above, is entirely standard.

There are two ways out of this problem for the Conservative, neither of them particularly attractive. The first is to make the move Levi makes above, to say that an agent should adopt a strategy for getting through a decision-tree and refuse to reconsider it at later stages. The above objections to that move still apply. The other move is to deny the following rule for reducing complex bets to simple bets.

Reduction: If $C(\beta, \chi) = \delta$ for any $\delta \in \{\beta, \chi\}$, then $C(\alpha, (\beta, \chi)) = C(\alpha, \delta)$.

To explain the notation, by $C(\alpha, \beta) = \alpha$ I mean that in a choice between holding α and trading it for β, it is rationally compelling that α be chosen. The underlining on α indicates that α is what is currently held; this is important because by the Conservative's lights $C(\alpha, \beta) = \alpha$ and $C(\alpha, \beta) = \beta$ is consistent. $C(\alpha, (\beta, \chi)) = \delta$ ($\delta \in \{\alpha, \beta, \chi\}$) means that an agent facing the choice between holding α and trading it for β with the knowledge that this can in turn be traded for χ will end up holding δ. Note that I don't assume $C(\alpha, \beta)$ is always defined.

I don't have any particularly strong arguments for *Reduction*, but it does have a high degree of intuitive plausibility. It is hard to see what other approach could be taken. If anyone thinks it is possible to justify avoiding *Reduction* and hence can avoid this problem I might not have much of a reply. I don't know of any such justification, and I can't see how it could be intuitively plausible, but I'm not going to try and write knock-down objections to as yet unformulated justifications.[4]

5. Caprice

To set out the correct decision-rule, Caprice, I need a new piece of terminology. Say ψ is *almost preferred* to ϕ according to P iff for all Pr in P, $E_{Pr}(\psi) \geq E_{Pr}(\phi)$. When no ambiguity results I omit the 'according to P. Clearly whenever ψ is strictly preferred to ϕ it is almost preferred, but the converse is not true. Unlike strict preference, almost preference is not anti-symmetric. Bets ψ and ϕ can each be almost preferred to the other.

The core idea behind Caprice is that there should be as few restrictions on rational choice as possible apart from the rule that, whenever ψ is strictly preferred to ϕ it is irrational to choose ϕ over ψ. Unfortunately, as

[4] Seidenfeld (1994) rejects the idea that $C(\alpha, (\beta, \chi)) = C(\alpha, \beta, \chi)$, i.e. the idea that the dynamic choice can be reduced to a three way static choice. He thinks it is plausible that $C(\alpha, (\beta, \chi)) = \alpha$ even when β has a higher payout than α on every possible state of affairs. But nevertheless he accepts, at page 459n, *Reduction*, which he calls 'backward induction', showing that the attractiveness of this principle is to a large extent independent of one's views on related principles concerning dynamic and static decisions, or equivalently concerning extensional and normal form decisions.

it is, this won't do, because it permits the following irrational course of action. Recall the earlier example where ϕ and ψ are almost indifferent, as are ϕ + \$5 and ψ. If there were no rational restrictions on trade between almost indifferent bets then there would be no grounds for criticising the trader who first swaps ϕ + \$5 for ψ and then swaps ψ for ϕ. Yet presumably it should be possible to subject this person to rational criticism.

I think the best thing to say about this case is that neither trade is itself irrational, but they are an irrational combination. In most decision-theories on the market this option is ruled out by stipulation; a set of trades is irrational iff one member of that set is irrational. There is, however, no reason to make such a restriction. Consider this analogy with belief. It seems plausible to say that it is reasonable to believe Oswald killed Kennedy and reasonable to believe he didn't, but it isn't reasonable to believe both that Oswald killed Kennedy and that he didn't. A set of beliefs, each reasonable on its own, might be unreasonable in combination. I claim we can say the same about decisions. A set of decisions, each reasonable on its own, might be unreasonable.

Because of this intuition, the Caprice rule must be expressed in terms of the reasonableness of sets of decisions. This can be applied easily to simple choices by looking at singleton sets. The notation $\#(\alpha, \beta) = \delta$ ($\delta \in \{\alpha, \beta\}$) means that δ is chosen (by the agent under consideration) in a pairwise choice between α and β. This is a different concept to the earlier $C(\alpha, \beta)$ notation in two respects. First, it is descriptive not normative. Given that I am usually discussing ideal agents this isn't as big a difference as it might normally seem. Secondly, $\#(\alpha, \beta)$ can be defined, even for rational agents, when $C(\alpha, \beta)$ is not. If α and β are almost indifferent, but when faced with the choice between them the agent chooses α, then $C(\alpha, \beta)$ is undefined (according to Caprice), but $\#(\alpha, \beta) = \alpha$.

> *Caprice*: A set *S* of choices of the form $\#(\alpha_i, \beta_i) = \alpha_i$ ($i \in \{1, ..., n, ...\}$) is rationally permissible according to P iff there is some non-empty subset G of P such that for all i, α_i is almost preferred to β_i according to G.

Caprice is only defined in terms of pairwise choices. If α is chosen in a three-way choice between α, β and χ, we say $\#(\alpha, \beta) = \alpha$ and $\#(\alpha, \chi) = \alpha$. This can easily be extended to *n*-way choices. Hence a single *n*-way choice, with $n > 2$, can be regarded as a many-element set of pairwise choices.

Note two immediate consequences of this rule. First, when we are just considering a single choice between almost indifferent bets ϕ and ψ, either choice is acceptable. In trading terms, it is permissible but not compelling to trade ϕ for ψ. This is the motivation for calling the rule 'Caprice'. Secondly, any set of choices which leaves the trader with a position such that they would strictly prefer to be back where they started is not rationally permissible according to Caprice. Hence Caprice as specified captures two important intuitive requirements on decision-rules.

I haven't yet specified how Caprice should be applied to choices between nodes of a decision-tree, because here there isn't much to say. In cases like that set out in Figure 1, the Capricious decision-maker can simply decide which end-point she wants to end up with, and follow the tree to that point. Provided her original *n*-way choice is permissible, every pair-wise choice she makes will be permissible. I showed above that the only way for the Conservative to avoid absurd decisions was to be closed-minded in the sense that she had to deliberately decide *not* to reflect at various stages in the tree about whether her initial strategy should be carried through. By comparison, the Capricious agent can be completely reflective.

There is one interesting special case of Caprice, which I'm adopting from Smets (1994). It isn't Smets's preferred approach for a couple of reasons, not least being that Smets advocates dropping conditionalisation, but the terminology and idea is largely his. An agent whose representor is P has P as their *credal* probability function. They arbitrarily select an element Pi from P to use for decision-making purposes; this is their *pignistic* probability. ('Pignistic' is from the Latin *pignus*, meaning to bet.) When making a choice between gambles they choose that gamble α such that $E_{Pi}(\alpha)$ is maximised. An agent who does this will never do anything wrong according to Caprice.

I noted at page 6 that any decision-rule would have to give up one of Arrow's constraints (1) through (4). Caprice gives up (2). It says that sometimes given the composition of P we simply can't say which of two bets should be chosen. If this pignistic approach is followed, in a sense (2) is kept at the cost of (3). The pignistic probability function becomes the dictator in Arrow's sense. This might be an improvement; I leave it up to the reader to decide whether or not it is.

There is one odd result as a consequence of adopting Caprice. An agent is told (reliably) that there are red and black marbles in a box in front of them, and a marble is to be drawn from the box. They are given the choice between three bets. α pays $1 if a red marble is drawn, nothing otherwise, β pays a certain 45 cents, and χ pays $1 if a black marble is drawn. Is it rationally permissible for the agent to choose β, again assuming constant marginal utility of money?

Levi (1974) writes as if it is obvious that choosing β is irrational. This is a cornerstone of the 'impeccable' analysis which leads to a dismissal of (4) but receives almost no justification. Jeffrey (1983) defines Bayesian approaches to decision-making so that choosing β is not Bayesian, but of course it isn't an obvious truth that only Bayesian approaches are correct. Dempster (1988) claims that choosing β is permissible, and perhaps even compelling, though it appears he is motivated by the maximin rule, which I showed above is flawed.

I only bring this up to note that Caprice says it is not rational to choose β. To see this, assume we choose β. We will now show that G must be empty. Let p be the proposition that the marble to be drawn is red. Since β is almost preferred to α according to G, for every Pr in G it follows that $Pr(p) \leq 0.45$. However, since β is almost preferred to χ according to G, for every Pr in G it follows that $Pr(p) \geq 0.55$. There is no Pr satisfying each of these constraints, hence G is empty. It doesn't however, appear at all intuitively compelling that it should be irrational to choose β. A defender of Caprice has to either explain away this intuition or, like Levi, simply deny that the intuition exists. The first of these choices is possible. One approach already noted is to say a choice of β reflects an irrational commitment to Maximin. Another is to say that it reflects a failure to internalise fully the assumption that the marginal utility of money is constant. I suspect that is what explains my intuition that β is an acceptable choice. I don't think this raises a huge problem for the defender of Caprice – some questions are always going to be spoils to the victor – but it is a little disconcerting. If there is to be a strong attack on Caprice, I suspect it will be built around cases like this one.

6. Arguments For Caprice

Apart from the fact that it avoids the pitfalls of its more well-known rivals, there are two positive arguments for Caprice. Each of them is essentially the reverse of an argument I used against Levi. I'll call them the arguments from Arrow and Buridan.

The argument from Arrow notes that the four principles Arrow gave, (1) to (4) above, are inconsistent. Hence we must give up one of them. As there is strong intuitive support for Pareto, Non-Dictatorship and Independence of Irrelevant Alternatives, it seems the correct decision theory must give up what Arrow calls 'Collective Rationality', but what is perhaps better called Completeness in our context. There must be some choices about which our decision theory is silent. Since Caprice, unlike its popular rivals, satisfies this constraint, this is something in its favour. Of course this is not an argument against other incomplete rivals of Caprice. However, one strength of Caprice is that the class of decisions over which it is silent is quite a natural class. I doubt there could be a smaller class than this which is equally natural.

This leads to the argument from Buridan. Given the way I have set out the problem, when ϕ and ψ are almost indifferent, there is no reason to choose one over the other. The agent really is in the position of Buridan's ass. Of course like the ass the agent may be well advised to choose either ϕ or ψ over some less attractive alternatives. Unlike all its rivals, Caprice takes this conclusion seriously. If there is no reason to choose ϕ over ψ or *vice versa*, there really is no reason. It doesn't go and say this and then find a reason.

In particular, it must be really inexplicable why an agent chooses ϕ over ψ or *vice versa* in such cases. Should there be such a reason, it must be traceable to the beliefs and desires (or more generally partial beliefs and preferences) of the agent. The assumption of incomparability is just the assumption that those beliefs and desires don't determine a choice. Hence any decision theory must agree with Caprice's 'no explanation' conclusion. Given this, it is hard to see how the correct theory can differ from Caprice.

It might be thought that Caprice breaches this 'no explanation' rule in an important case. Say the expected value of ϕ is vague over [$30, $40], and that an agent has just sold a unit of ϕ for $32. According to Caprice, if she now buys a unit for $38, or indeed any price over $32, she will have acted irrationally. Does this mean that either (i) the value of ϕ is now vague over merely [$30, $32] or (ii) the value is unchanged but she now has a reason for not buying ϕ for more than $32? According to the objection, I have ruled out (i) and (ii), but I am committed to one of them.

The objection is in part correct, I have ruled out (i) and (ii). However, I am not committed to their disjunction. Were the agent to now buy ϕ for $38, that would not of itself be an irrational act, however it would take her from having performed a set of rational acts to having performed a set of irrational ones. The only reason one would think this implies the last act is irrational is if one was wedded to the idea that a set of acts is irrational iff it includes an irrational act. By that principle, an agent can only move from a rational to an irrational set by performing an irrational act. However, that is a principle I gave reasons for rejecting in setting out Caprice.

Again the analogy with belief is instructive. If the agent believed yesterday that Oswald killed Kennedy, she can't rationally believe today that Oswald didn't kill Kennedy unless she ceases to believe that he did kill him. But, and here's the difference, yesterday's beliefs can be more easily undone than yesterday's trades. If she could cease to have sold ϕ for $32 yesterday, she can rationally buy it for $38 today. Sometimes this will be possible (if the sale has a 'cooling off' period), but usually it will be just as fixed as the rest of the past. It is because she can change her beliefs, but not her trades, that we judge an agent's trades diachronically, but her beliefs largely synchronically. When we keep all this in mind, we won't unduly focus on her last trade and judge it too harshly.

7. Summary

Most of the attempts to formulate a vague decision theory have been attempts to say what ought be done when faced with almost indifferent options. The decision theory here says that we cannot consistently answer this question. For one thing, it is incoherent to say the options are almost indifferent and on the other hand that we can decide between them. This is just an admission that we have not included all the relevant material in our representation of the agent's beliefs and desires. Secondly, the attempts to answer the question so far have all led to dynamic incoherence, and Arrow's Theorem suggests this is unavoidable. So it is best to settle for the minimal constraints supplied by Caprice.[5]

References

Arntzenius, Frank and David McCarthy (1997) "The Two-Envelope Paradox and Infinite Expectations" *Analysis* 47: 42-50.

Arrow, Kenneth (1963) *Social Choice and Individual Values.* New York: Wiley.

Dempster, Arthur (1988) "Probability, Evidence and Judgment" (with discussion) in D. Bell, H. Raiffa and A. Tversky (eds) *Decision Making*. Cambridge: Cambridge University Press. pp 283-298.

Gilboa, Itzhak and David Schmeidler (1993) "Updating Ambiguous Beliefs" *Journal of Economic Theory* 59: 33-49.

Hart, A. G. (1942) "Risk, Uncertainty and the Unprofitability of Compounding Probabilities" in O. Lange, F. McIntyre and T. O. Yntema (eds) *Studies in Mathematical Economics and Econometrics*. Chicago: University of Chicago Press.

Hausman, Daniel (1991) *The Inexact and Separate Science of Economics*. Cambridge: Cambridge University Press.

Hájek, Alan (to appear) Objecting Vaguely to Pascal's Wager, manuscript.

Jaffray, J. (1994) Decision Making with Belief Functions. In Yager, Fedrizzi and Kacpzyk (eds) pp. 331-352.

Jeffrey, Richard (1983) "Bayesianism with a Human Face", in J. Earman (ed.) *Testing Scientific Theories*, Minneapolis: University of Minnesota Press.

Kyburg, Henry (1961) *Probability and the Logic of Rational Belief*. Middletown: Wesleyan University Press.

Kyburg, Henry (1974) *The Logical Foundations of Statistical Inference*. Dordrecht: Reidel.

Kyburg, Henry (1983) *Epistemology and Inference*. Minneapolis: University of Minnesota Press.

Levi, Isaac (1974) "On Indeterminate Probabilities" *Journal of Philosophy* 71: 391-418.

Levi, Isaac (1980) *The Enterprise of Knowledge*. Cambridge, MA.: MIT Press.

[5] Thanks to Alan Hájek and Graham Oppy for many helpful comments and suggestions on earlier drafts of this paper.

Levi, Isaac (1986) *Hard Choices*. Cambridge: Cambridge University Press.

Levi, Isaac. (1987) "The Demons of Decision" *Monist* 70: 193 - 211.

Lewis, David (1981) "Causal Decision Theory" *Australasian Journal of Philosophy* 59: 5-30.

Norton, John (1998) "When the Sum of Our Expectations Fails Us" *Pacific Philosophical Quarterly* 79: 34-58.

Ramsey, Frank (1926) "Truth and Probability" in his (1931), pp 156-198.

Seidenfeld, Teddy (1984) Review of Paul Horwich's *Probability and Evidence*. *Philosophical Review* 93 474-483.

Seidenfeld, Teddy (1994) "When Normal and Extensive Form Decisions Differ" in *Logic, Methodology and Philosophy of Science*, edited by Dag Prawitz, Brian Skyrms and Dag Westerståhl, Amsterdam: Elsevier, pp 451-463.

Smets, Phillippe (1994) "What is Dempster-Shafer's Theory of Evidence?" In In Yager, Fedrizzi and Kacpzyk (eds) *Advances in the Dempster-Shafer Theory of Evidence*. New York, NY.: John Wiley, pp 5-34.

Smith, Cedric A. B. (1961) "Consistency in Statistical Inference and Decision" *Journal of the Royal Statistical Society* Series B 23: 1 - 37.

Strat, T. (1990) "Decision analysis using Belief Functions" *International Journal of Approximative Reasoning* 4: 391-417.

Tintner, Gerhard (1941) "The Theory of Choice Under Subjective Risk and Uncertainty" *Econometrica* 9: 298-304.

van Fraassen, Bas (1990) "Figures in a Probability Landscape" in J. Dunn and A. Gupta (eds) *Truth or Consequences*. Amsterdam: Kluwer, pp 345-356.

van Fraassen, Bas (1995) "Belief and the Problem of Ulyssess and the Sirens" *Philosophical Studies* 77: 7 - 37.

Walley, Peter (1991) *Statisical Reasoning with Imprecise Probabilities* London: Chapman & Hall.

Wang, Hao (1987) *Reflections on Gödel*. Cambridge, MA: MIT Press.

Williams, Peter (1976) Indeterminate Probabilities. In M. Przelecki, K. Szaniawski and R. Wojcicki (eds) Formal *Methods in the Methodology of the Empriccal Sciences*. Dordrecht: Reidel, pp 229-246.

Williams, Peter (1978) "On a New Theory of Epistemic Probability" *British Journal for the Philosophy of Science* 29: 375-387.